1 Eigen Value *(15 points)*

Suppose $\lambda_1, \lambda_2, ..., \lambda_n$ are eigen values of matrix A Prove:

- $\lambda_1 + \lambda_2 + ... + \lambda_n = trace(A)$.
- $\lambda_1 \lambda_2 ... \lambda_n = \det(A)$.
- AB and BA have the same set of eigen values.
- A and A^T have the same set of eigen values.

2 Covariance & Expectation *(15 points)*

If we define $cov(X, Y|Z)$ as below, prove the following equations.

$$cov(X, Y|Z) = E[(X - E[X|Z])(Y - E[Y|Z])|Z]$$

- $Var(X) = Var(E[X|Y]) + E(Var[X|Y])$
- $cov(X, Y|Z) = E[XY|Z] - E[X|Z]E[Y|Z]$
- $cov(X, Y) = E[cov(X, Y|Z)] + cov(E[X|Z], E[Y|Z])$

3 Matrix Derivative *(14 points)*

- $\frac{\partial x^T A x}{\partial x} = 2x^T A$
- $\frac{\partial trace(X^T A X)}{\partial X} = X^T(A + A^T)$

4 Random Variable *(10 points)*

Assume X as a random variable from uniform distribution $U\ (0, 1)$. Compute PDF for $\sqrt{X}$ & X^2.

5 Rank *(8 points)*

Prove that if P is a full rank matrix, matrices M and $P^{-1}MP$ have the same set of eigen values.

6 Matrix Factorization *(14 points)*

- Prove that every symmetric positive definite matrix A has a unique factorization of the form $A = LL^T$, where L is a lower triangular matrix with positive diagonal entries.
- Suppose A is a matrix and we have $A = QR$ (Q is an orthonormal matrix). in this case, SVD of matrix A will be look like the SVD of matrix R; however, they will not be identical. Find the difference between these two SVD factorizations.

7 Nilpotent matrix *(10 points)*

square matrix such as N is a nilpotent matrix if we can find k, which is a positive integer, that fits in the equation below.

$$N^k = 0$$

The smallest k that fits in this equation is called the index of N.
prove that if A is a nilpotent matrix with index k, $I - A$ is an invertible matrix then find its inversion.

8 MAP *(14 points)*

Suppose X, Y are independent normal random variables with mean μ and variance 1, where $\mu \sim Uni(0, 1)$.

$$f_\mu(t) = \begin{cases} 1 & t \ \in [0, 1] \\ 0 & o.w \end{cases}$$

- Find the joint distribution of μ, X, Y. ($f_{\mu,X,Y}$(t,x,y)).
- Find the MAP estimate of μ.