1 (10 points)

1.1

Explain the relation between k-means and EM algorithm. Can these two algorithms be the same? If yes, under what conditions and if not, why? (2 points)

1.2

Consider a density model given by a mixture distribution as below:

$$p(x) = \sum \pi_k p(x|k)$$

Suppose that the vector x is partitioned into two parts so that x = (x_a, x_b). Show that the conditional density $P(x_a|x_b)$ is itself a mixture distribution. Find expressions for the mixing coefficients (π_is) and for the component densities in terms of p. (4 points)

1.3

Consider a special case of a Gaussian mixture model in which the covariance matrices Σ_k of the components are all constrained to have a common value Σ. Derive the EM equations for maximizing the likelihood function (write the M step). (4 points)

2 (15 points)

Consider the below mixture distribution in a way that each mixture component cames from a poisson distribution with parameter θ_i:

$$p(x) = \sum_{i=1}^{I} \pi_i p_i(x)$$

2.1

Write expectation and maximization steps to estimate the mixture distribution. (5 points)

2.2

Suppose that data samples are independent and identically distributed. Write down lagrange multiplier to maximize the lower bound with respect to Q. (10 points)

3 (15 points)

Suppose K gamblers that on each day t, one of the K gamblers plays m_t rounds of game and wins w_t of those m_t rounds. You just know the number of gamblers, the total number of played rounds in each day and the number of rounds won by the gambler. But you don't know which of the K gamblers played on which day. You decide to use mixture model to model this problem. For each player k, you model the probability that she wins on any given round by p_k. Hence, on day t, if the k'th gambler had played m_t rounds, then the probability that she won w_t rounds is given by a binomial distribution.(I)
You shall use a mixture of K binomials with parameters p_1, ..., p_K to model the data for n days given by (m_n, w_n), ... , (m_t, w_t). That is, on day t, we first pick one gambler out of the K at random according to the distribution π as $c_t \sim \pi$. Next, the gambler for day t plays m_t rounds and given the gambler is c_t, the number of wins w_t out of the m_t rounds is given by a binomial distribution. (II)

3.1

Write down distributions described in (I) and (II). (3 points)

3.2

Write down the E-step update for Q in terms of parameters from previous iteration (Write what $Q_t^{(i)}[k]$ is in each iteration i). (5 points)

3.3

For any, mixture models, the M-step for π on iteration i is calculated from this equation:

$$\pi^i[k] = \frac{\sum_{t=1}^{n} Q_t^{(i)}[k]}{n}.$$

Derive the M-step update for $p_1^{(i)}, ..., p_K^{(i)}$ the K model parameters on iteration i, in terms of data and $Q_t^{(i)}$'s. First write down the maximization problem for the M-step and then solve for p $p_1^{(i)}, ..., p_K^{(i)}$ showing that they are the maxima for optimization problem. (7 points)

4 (20 points)

Consider X and θ as the symbols for observed samples and parameters respectively.

4.1

Suppose that the distribution of parameters is p(θ). Write the expectation and maximization steps to maximize p($\theta|X$). (5 points)

4.2

Let X be a random variable with 4 states in it's sample space. Suppose that θ is a real number in [0,1] and each of these 4 states has the probability as follows:

State	Probability
A	$\frac{1}{3}$
B	$\frac{1}{3}(1-\theta)$
C	$\frac{2}{3}(\theta)$
D	$\frac{1}{3}(1-\theta)$

Suppose n experiments are done on X and A, B, C, D are observed n_a, n_b, n_c and n_d times respectively. Unfortunately, the value of n_a and n_c remained unknown. Assuming that θ has below distribution initially, write E and M steps to find θ. (15 points)

$$p(\theta) = \frac{\Gamma(v_1 + v_2)}{\Gamma(v_1)\Gamma(v_2)} \theta^{v_1 - 1} (1-\theta)^{v_2 - 1}$$

5 Practical (40 points)

Choose an image and implement GMM to segment it.
Implement E and M steps of the image segmentation process by your own.
For better results, first implement kmeans algorithm or investigate the image histogram to decide about the number of classes and other initial variables. Explain how you set your initial number of classes, means, variances, ...
Upload your original image and result image in a .jpg format, too.